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2002 J. Phys. A: Math. Gen. 35 6619

(http://iopscience.iop.org/0305-4470/35/31/307)

J. Phys. A: Math. Gen. **35** (2002) 6619–6629 PII: S0305-4470(02)36560-0

Conformal compactification of spacetimes

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Received 2 May 2002
Published 26 July 2002
Online at stacks.iop.org/JPhysA/35/6619

Abstract

The conformal groups for the nine two-dimensional real spaces of constant curvature are realized as matrix groups acting as globally defined linear transformations in a four-dimensional 'conformal ambient space'. This affords a unified and global study of the 'conformal completion' or compactification for the three classical Riemannian spaces as well as of the six relativistic and non-relativistic spacetimes (Minkowskian, de Sitter, anti-de Sitter, both Newton–Hooke and Galilean). The conformal embedding of the initial space into its compactification is carried out explicitly through two methods: either a group-theoretical one involving one-parameter subgroups or a geometric one by means of a stereographic projection. In the Euclidean and Minkowskian spaces the results reduce to the well known ones, but in the generic situation, with any non-zero curvature or arbitrary type signature, the approach is very explicit and provides some new insights.

PACS numbers: 02.20.−a, 02.40.−k, 11.25.Hf

1. Introduction

In a previous paper [1] we have developed a new approach to the study of cycle-preserving (conformal) transformations of the three 2D Riemannian spaces and the six $(1+1)$D spacetimes with constant curvature, which provides explicit expressions for the vector fields generating cycle-preserving transformations. This has been accomplished within a Cayley–Klein (CK) framework [2–6], which is able to produce general expressions, holding simultaneously for the nine spaces which are parametrized in terms of two parameters κ_1, κ_2: their constant curvature is κ_1 and metric signature type is $(+1, \kappa_2)$. The CK spaces $S^2_{[\kappa_1],\kappa_2}$ arise for the nine essentially different sets of particular values of κ_i (see table 1). We recall that in spacetimes: $\kappa_1 = \pm 1/\tau^2$ and $\kappa_2 = -1/c^2$, where τ is the universe time radius and c is the speed of light.

The conformal Lie algebra so obtained, $\mathrm{conf}_{\kappa_1,\kappa_2}$, is spanned by generators of translations P_i along two orthogonal directions (which whenever $\kappa_2 \leqslant 0$ can be seen as time and space

0305-4470/02/316619+11$30.00

directions), rotations (or boosts) J_{12}, specific conformal transformations G_i and dilations D ($i = 1, 2$), and their commutation rules read

$$\begin{array}{lll}
[J_{12}, P_1] = P_2 & [J_{12}, P_2] = -\kappa_2 P_1 & [P_1, P_2] = \kappa_1 J_{12} \\
[J_{12}, G_1] = G_2 & [J_{12}, G_2] = -\kappa_2 G_1 & [G_1, G_2] = 0 \\
[D, P_i] = P_i + \kappa_1 G_i & [D, G_i] = -G_i & [D, J_{12}] = 0 \\
[P_1, G_1] = D & [P_2, G_2] = \kappa_2 D & \\
[P_1, G_2] = -J_{12} & [P_2, G_1] = J_{12}. &
\end{array} \tag{1.1}$$

The conformal algebra $\mathrm{conf}_{\kappa_1,\kappa_2}$ is isomorphic to $so(3,1)$, $iso(2,1)$ and $so(2,2)$ according to $\kappa_2 >, =, < 0$, respectively (regardless of the value of κ_1). The three generators $\{J_{12}, P_1, P_2\}$ close the initial CK algebra $so_{\kappa_1,\kappa_2}(3)$, that is, the Lie algebra of isometries of these spaces.

As is well known, such a Lie algebra approach only provides a realization of the conformal Lie group $\mathrm{conf}_{\kappa_1,\kappa_2}$ as a *local* group of transformations in the CK space $S^2_{[\kappa_1],\kappa_2}$. The need to suitably complete $S^2_{[\kappa_1],\kappa_2}$ with 'additional' points in order to have conformal transformations defined as a *global* group of transformations arises because the vector fields so obtained and closing the Lie algebra (1.1) are, in general, not complete. An example is provided by the hyperbolic space $\mathbf{H}^2 \equiv S^2_{[-],+}$, for which the finite one-parameter transformations generated by the dilation D are given, in polar coordinates, by [1]:

$$\tanh(r'/2) = \mathrm{e}^{\lambda} \tanh(r/2) \tag{1.2}$$

where e^{λ} is a 'similarity factor'. A dilation with centre O and factor $\mathrm{e}^{\lambda} > 1$ transforms the interior of a disc of centre O and radius $r = 2\,\mathrm{argtanh}(\mathrm{e}^{-\lambda})$ into the *entire* hyperbolic space, because $r' = \infty$. Therefore, in order to have a global group, the images of the points *outside* the disc should be added to the 'ordinary' points, as otherwise these points cannot have an image by the dilation.

In this paper we propose a new description, covering the nine CK spaces in the same run, of their corresponding *inversive* or *conformal compactifications*, where cycle-preserving transformations can be defined as *global transformations*. In section 2 the 2D conformal group is realized as a 4×4 matrix Lie group providing a linear action in a 4D conformal ambient space. In section 3 the conformal completion or compactification of the CK space $S^2_{[\kappa_1],\kappa_2}$ is defined as a homogeneous space $\mathrm{conf}_{\kappa_1,\kappa_2}/\mathrm{sim}_{0,\kappa_2}$, where $\mathrm{sim}_{0,\kappa_2}$ is a subgroup of $\mathrm{conf}_{\kappa_1,\kappa_2}$ isomorphic to the similitude subgroup of the *flat* CK space $S^2_{[0],\kappa_2}$. The *conformal embedding* of the initial CK space into its conformal compactification is performed in section 4 by means of two different procedures. The former makes use of one-parameter conformal subgroups parametrizing the conformal space in terms of geodesic coordinates of the CK space. The latter is a stereographic projection of the CK space into its conformal completion. These results extend to all the nine CK spaces the conformal compactification and linearization already familiar for the Euclidean [7, 8] and Minkowskian spaces [7, 9–12] yet it does this extension in a way which still makes sense for curved and/or degenerate metric spaces. A discussion of the conformal completion for each particular space is also performed. Some comments close the paper.

2. Conformal groups

Let us consider the following 4×4 real matrix representation of the conformal algebra $\mathrm{conf}_{\kappa_1,\kappa_2}$ (1.1), which can be obtained from its identification with $so(3,1)$, $iso(2,1)$ or

$so(2,2)$ according to the sign of κ_2 [13]:

$$P_1 = \frac{1}{2\ell}\begin{pmatrix} 0 & 0 & -1-\kappa_1\ell^2 & 0 \\ 0 & 0 & -1+\kappa_1\ell^2 & 0 \\ 1+\kappa_1\ell^2 & -1+\kappa_1\ell^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad J_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\kappa_2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$P_2 = \frac{1}{2\ell}\begin{pmatrix} 0 & 0 & 0 & -\kappa_2(1+\kappa_1\ell^2) \\ 0 & 0 & 0 & -\kappa_2(1-\kappa_1\ell^2) \\ 0 & 0 & 0 & 0 \\ 1+\kappa_1\ell^2 & -1+\kappa_1\ell^2 & 0 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.1}$$

$$G_1 = \ell\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad G_2 = \ell\begin{pmatrix} 0 & 0 & 0 & \kappa_2 \\ 0 & 0 & 0 & -\kappa_2 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}.$$

As the product $\kappa_1\ell^2$ is dimensionless, the parameter ℓ is a non-zero constant with dimensions of *length*; when $\kappa_2 < 0$ this will be considered as a 'time-like' length. Note that P_i and G_i must be dimensionally inverse to each other [14].

Exponentiation gives rise to the following one-parametric subgroups of $\mathrm{conf}_{\kappa_1,\kappa_2}$, where we have introduced the curvature-dependent trigonometric functions: cosine $C_\kappa(x)$, sine $S_\kappa(x)$ and versed sine $V_\kappa(x)$, fully described in [1, 4]:

$$\mathrm{e}^{\mu_1 P_1} = \begin{pmatrix} 1-\frac{(1+\kappa_1\ell^2)^2}{4\ell^2}V_{\kappa_1}(\mu_1) & \frac{(1-\kappa_1^2\ell^4)}{4\ell^2}V_{\kappa_1}(\mu_1) & -\frac{(1+\kappa_1\ell^2)}{2\ell}S_{\kappa_1}(\mu_1) & 0 \\ -\frac{(1-\kappa_1^2\ell^4)}{4\ell^2}V_{\kappa_1}(\mu_1) & 1+\frac{(1-\kappa_1\ell^2)^2}{4\ell^2}V_{\kappa_1}(\mu_1) & -\frac{(1-\kappa_1\ell^2)}{2\ell}S_{\kappa_1}(\mu_1) & 0 \\ \frac{(1+\kappa_1\ell^2)}{2\ell}S_{\kappa_1}(\mu_1) & -\frac{(1-\kappa_1\ell^2)}{2\ell}S_{\kappa_1}(\mu_1) & C_{\kappa_1}(\mu_1) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{e}^{\mu_2 P_2} = \begin{pmatrix} 1-\frac{\kappa_2(1+\kappa_1\ell^2)^2}{4\ell^2}V_{\kappa_1\kappa_2}(\mu_2) & \frac{\kappa_2(1-\kappa_1^2\ell^4)}{4\ell^2}V_{\kappa_1\kappa_2}(\mu_2) & 0 & -\frac{\kappa_2(1+\kappa_1\ell^2)}{2\ell}S_{\kappa_1\kappa_2}(\mu_2) \\ -\frac{\kappa_2(1-\kappa_1^2\ell^4)}{4\ell^2}V_{\kappa_1\kappa_2}(\mu_2) & 1+\frac{\kappa_2(1-\kappa_1\ell^2)^2}{4\ell^2}V_{\kappa_1\kappa_2}(\mu_2) & 0 & -\frac{\kappa_2(1-\kappa_1\ell^2)}{2\ell}S_{\kappa_1\kappa_2}(\mu_2) \\ 0 & 0 & 1 & 0 \\ \frac{(1+\kappa_1\ell^2)}{2\ell}S_{\kappa_1\kappa_2}(\mu_2) & -\frac{(1-\kappa_1\ell^2)}{2\ell}S_{\kappa_1\kappa_2}(\mu_2) & 0 & C_{\kappa_1\kappa_2}(\mu_2) \end{pmatrix}$$

$$\mathrm{e}^{\nu_1 G_1} = \begin{pmatrix} 1-\frac{1}{2}\nu_1^2\ell^2 & -\frac{1}{2}\nu_1^2\ell^2 & \nu_1\ell & 0 \\ \frac{1}{2}\nu_1^2\ell^2 & 1+\frac{1}{2}\nu_1^2\ell^2 & -\nu_1\ell & 0 \\ -\nu_1\ell & -\nu_1\ell & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{e}^{\psi J_{12}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & C_{\kappa_2}(\psi) & -\kappa_2 S_{\kappa_2}(\psi) \\ 0 & 0 & S_{\kappa_2}(\psi) & C_{\kappa_2}(\psi) \end{pmatrix}$$

$$\mathrm{e}^{\nu_2 G_2} = \begin{pmatrix} 1-\frac{1}{2}\kappa_2\nu_2^2\ell^2 & -\frac{1}{2}\kappa_2\nu_2^2\ell^2 & 0 & \kappa_2\nu_2\ell \\ \frac{1}{2}\kappa_2\nu_2^2\ell^2 & 1+\frac{1}{2}\kappa_2\nu_2^2\ell^2 & 0 & -\kappa_2\nu_2\ell \\ 0 & 0 & 1 & 0 \\ -\nu_2\ell & -\nu_2\ell & 0 & 1 \end{pmatrix} \qquad \mathrm{e}^{\xi D} = \begin{pmatrix} \cosh\xi & \sinh\xi & 0 & 0 \\ \sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

In this representation the conformal group $\mathrm{conf}_{\kappa_1,\kappa_2}$ acts in a *conformal linear ambient space* $\mathbb{R}^4 = (s^+, s^-, s^1, s^2)$ as the group of linear isometries of a bilinear form

$$\Upsilon = \mathrm{diag}(1, -1, 1, \kappa_2) \tag{2.3}$$

that is, any element $X \in \mathrm{conf}_{\kappa_1,\kappa_2}$ fulfils

$$X^T \Upsilon X = \Upsilon \tag{2.4}$$

where X^T denotes the transpose matrix of X. Therefore the action of $\mathrm{conf}_{\kappa_1,\kappa_2}$ preserves the quadratic form $(s^+)^2 - (s^-)^2 + (s^1)^2 + \kappa_2(s^2)^2$. The subgroup spanned by $\{J_{12}, G_1, G_2\}$ leaves invariant the *origin* point $\mathcal{O} = (1, -1, 0, 0) \in \Gamma_0$, while the dilation subgroup $\exp(\xi D)$ transforms $\mathcal{O}$ into $(\mathrm{e}^{-\xi}, -\mathrm{e}^{-\xi}, 0, 0)$. Consequently, $\langle J_{12}, G_1, G_2, D\rangle$ spans the isotropy subgroup of the *ray* of $\mathcal{O}$, which turns out to be isomorphic to the similitude group of a 2D flat space: either the Euclidean ($\kappa_2 > 0$), Galilean ($\kappa_2 = 0$), or Poincaré ($\kappa_2 < 0$) similitude group; this is therefore denoted $\mathrm{sim}_{0,\kappa_2}$. The two remaining subgroups generated by P_1 and P_2 move the ray of $\mathcal{O}$. The orbit of $\mathcal{O}$ under $\mathrm{conf}_{\kappa_1,\kappa_2}$ is henceforth contained in the *cone* Γ_0 given by

$$\Gamma_0 \equiv (s^+)^2 - (s^-)^2 + (s^1)^2 + \kappa_2(s^2)^2 = 0. \tag{2.5}$$

3. Completed conformal spaces

The former description allows us to define, for any of the nine CK spaces $S^2_{[\kappa_1],\kappa_2}$, the corresponding 2D completed conformal space $C^2_{[\kappa_1],\kappa_2}$, also called *inversive CK space*, as a homogeneous space:

$$C^2_{[\kappa_1],\kappa_2} := \mathrm{conf}_{\kappa_1,\kappa_2}/\mathrm{sim}_{0,\kappa_2} \tag{3.1}$$

where $\mathrm{sim}_{0,\kappa_2}$ is a subgroup of $C^2_{[\kappa_1],\kappa_2}$, generated by J_{12}, D, G_1, G_2. Its structure is a semidirect product:

$$\mathrm{sim}_{0,\kappa_2} = T_2 \odot \big(SO_{\kappa_2}(2) \otimes SO(1,1)\big)$$
$$T_2 = \langle G_1, G_2\rangle \qquad SO_{\kappa_2}(2) = \langle J_{12}\rangle \qquad SO(1,1) = \langle D\rangle. \tag{3.2}$$

To get an explicit model of the conformal space, we need to consider in closer detail the orbit of the ray $\mathcal{O}$ under the action of the group $\mathrm{conf}_{\kappa_1,\kappa_2}$. Consider first the rays in this orbit (hence fulfilling (2.5)) for which $s^- \neq 0$; each such ray determines a unique and well-defined point in the section of Γ_0 by the hyperplane $s^- = -1$. The natural coordinates $(\tilde{s}^+, \tilde{s}^1, \tilde{s}^2)$ on this section are defined by $\tilde{s}^i = -s^i/s^-$ and verify

$$(\tilde{s}^+)^2 + (\tilde{s}^1)^2 + \kappa_2(\tilde{s}^2)^2 = 1 \longleftrightarrow S^2_{[+],\kappa_2} \tag{3.3}$$

displaying the identification of the set of these rays with the orbit of $\mathcal{O} \equiv (1, 0, 0)$ in a CK ambient space sphere for the CK constants $\kappa_1 = +1, \kappa_2$, hence to a 2D CK space with *positive* curvature $S^2_{[+],\kappa_2}$ (see table 1). By taking into account the sign of κ_2 we find after (2.5) two different situations:

- In Riemannian spaces with $\kappa_2 > 0$, the cone Γ_0 *cannot* contain directions with $s^- = 0$, that is, the set of rays in Γ_0 can directly be identified with the section $s^- = -1$ (3.3).
- In spacetimes with $\kappa_2 \leqslant 0$, the cone will *always* contain rays with $s^- = 0$. These rays will have no proper intersection with the section $s^- = -1$. However, these will appear as *points at infinity* in (3.3). Further, to deal with rays implies that these points should be identified through the ordinary antipodal identification in $\mathbb{R}^4$, which when $s^- = 0$ translates into antipodal identification in the section $s^- = -1$. Consequently we state the following.

Table 1. The nine 2D CK spaces $S^2_{[\kappa_1],\kappa_2} = SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2)$ and their conformal compactifications $C^2_{[\kappa_1],\kappa_2}$; we denote $ISO(1) = \mathbb{R}$ and NH means Newton–Hooke.

The three Riemannian spaces with $\kappa_2 > 0$:		
Elliptic: $\mathbf{S}^2$	Euclidean: $\mathbf{E}^2$	Hyperbolic: $\mathbf{H}^2$
$S^2_{[+],+} = SO(3)/SO(2)$	$S^2_{[0],+} = ISO(2)/SO(2)$	$S^2_{[-],+} = SO(2,1)/SO(2)$
have as their conformal compactification:		
$C^2_{[\kappa_1],+} \equiv \mathrm{conf}_{\kappa_1,+}/\mathrm{sim}_{0,+} \equiv SO(3,1)/(T_2 \odot (SO(2) \otimes SO(1,1))) \equiv S^2_{[+],+} \equiv \mathbf{S}^2$		
The three non-relativistic spacetimes with $\kappa_2 = 0$:		
Oscillating NH: $\mathbf{NH}^{1+1}_+$	Galilean: $\mathbf{G}^{1+1}$	Expanding NH: $\mathbf{NH}^{1+1}_-$
$S^2_{[+],0} = ISO(2)/ISO(1)$	$S^2_{[0],0} = IISO(1)/ISO(1)$	$S^2_{[-],0} = ISO(1,1)/ISO(1)$
have as their conformal compactification:		
$C^2_{[\kappa_1],0} \equiv \mathrm{conf}_{\kappa_1,0}/\mathrm{sim}_{0,0} \equiv ISO(2,1)/(T_2 \odot (ISO(1) \otimes SO(1,1))) \equiv \widetilde{S}^2_{[+],0} \equiv \widetilde{\mathbf{NH}}^{1+1}_+$		
The three relativistic spacetimes with $\kappa_2 < 0$:		
Anti-de Sitter: $\mathbf{AdS}^{1+1}$	Minkowskian: $\mathbf{M}^{1+1}$	De Sitter: $\mathbf{dS}^{1+1}$
$S^2_{[+],-} = SO(2,1)/SO(1,1)$	$S^2_{[0],-} = ISO(1,1)/SO(1,1)$	$S^2_{[-],-} = SO(2,1)/SO(1,1)$
have as their conformal compactification:		
$C^2_{[\kappa_1],-} \equiv \mathrm{conf}_{\kappa_1,-}/\mathrm{sim}_{0,-} \equiv SO(2,2)/(T_2 \odot (SO(1,1) \otimes SO(1,1))) \equiv \widetilde{S}^2_{[+],-} \equiv \widetilde{\mathbf{AdS}}^{1+1}$		

Theorem 1. *The conformal or inversive completion $C^2_{[\kappa_1],\kappa_2}$ of any of the nine 2D CK spaces can be described as*

$$C^2_{[\kappa_1],\kappa_2} \equiv \widetilde{S}^2_{[+],\kappa_2} := S^2_{[+],\kappa_2} \cup \{\textit{points at infinity in } S^2_{[+],\kappa_2} \textit{with antipodal identification}\}$$

and the conformal space $C^2_{[\kappa_1],\kappa_2}$ so obtained is always compact.

In this way, we find explicitly the completed conformal spaces for the nine CK spaces as displayed in table 1:

- The conformal completion of the three *Riemannian spaces* with $\kappa_2 > 0$ and any curvature κ_1 is the ordinary *sphere* $\mathbf{S}^2$. There are no points at infinity in $\mathbf{S}^2$, so that $\widetilde{\mathbf{S}}^2 = \mathbf{S}^2$.
- The three *non-relativistic spacetimes* with $\kappa_2 = 0$ ($c = \infty$) and any κ_1 have as their conformal compactification the space $\widetilde{\mathbf{NH}}^{1+1}_+$, obtained from the *oscillating Newton–Hooke* spacetime through antipodal identification of its points at infinity.
- The three *relativistic spacetimes* with $\kappa_2 = -1/c^2 < 0$ and any κ_1 have as their conformal compactification the space $\widetilde{\mathbf{AdS}}^{1+1}$, obtained from the *anti-de Sitter* spacetime through antipodal identification of its points at (spatial) infinity.

Furthermore, the relation (3.3) between the conformal space and a CK space suggests calling $(\tilde{s}^+, \tilde{s}^1, \tilde{s}^2)$ *conformal Weierstrass coordinates* of $C^2_{[\kappa_1],\kappa_2}$, which in turn can be parametrized in terms of three sets of *conformal geodesic coordinates*, similar to the CK spaces [1]; these are parallel I (A, Y), parallel II (X, B) and polar (R, Φ) coordinates:

$$\begin{aligned} \tilde{s}^+ &= \cos A\, C_{\kappa_2}(Y) = \cos X\, C_{\kappa_2}(B) = \cos R \\ \tilde{s}^1 &= \sin A\, C_{\kappa_2}(Y) = \sin X = \sin R\, C_{\kappa_2}(\Phi) \\ \tilde{s}^2 &= S_{\kappa_2}(Y) = \cos X\, S_{\kappa_2}(B) = \sin R\, S_{\kappa_2}(\Phi). \end{aligned} \tag{3.4}$$

Table 2. The conformal embedding $S^2_{[\kappa_1],\kappa_2} \longmapsto C^2_{[\kappa_1],\kappa_2}$ in three geodesic coordinate systems. Coordinates (s^+, s^-, s^1, s^2) parametrize the cone $\Gamma_0 \in \mathbb{R}^4$, while $(\tilde{s}^+, \tilde{s}^- = -1, \tilde{s}^1, \tilde{s}^2)$ are a parametrization of the conformal space itself $C^2_{[\kappa_1],\kappa_2} \equiv \widetilde{S}^2_{[+],\kappa_2}$. We have introduced the shorthand notation $\eta_+ = \frac{1+\kappa_1\ell^2}{2\kappa_1\ell^2}$ and $\eta_- = \frac{1-\kappa_1\ell^2}{2\kappa_1\ell^2}$.

Parallel I coordinates	Parallel II coordinates	Polar coordinates
$(a, y) \in S^2_{[\kappa_1],\kappa_2} \to (A, Y) \in C^2_{[\kappa_1],\kappa_2}$	$(x, b) \in S^2_{[\kappa_1],\kappa_2} \to (X, B) \in C^2_{[\kappa_1],\kappa_2}$	$(r, \phi) \in S^2_{[\kappa_1],\kappa_2} \to (R, \Phi) \in C^2_{[\kappa_1],\kappa_2}$
$s^\pm = \pm 1 - \eta_\pm\{1 - C_{\kappa_1}(a)C_{\kappa_1\kappa_2}(y)\}$	$s^\pm = \pm 1 - \eta_\pm\{1 - C_{\kappa_1}(x)C_{\kappa_1\kappa_2}(b)\}$	$s^\pm = \pm 1 - \eta_\pm(1 - C_{\kappa_1}(r))$
$s^1 = S_{\kappa_1}(a)C_{\kappa_1\kappa_2}(y)/\ell$	$s^1 = S_{\kappa_1}(x)/\ell$	$s^1 = S_{\kappa_1}(r)C_{\kappa_2}(\phi)/\ell$
$s^2 = S_{\kappa_1\kappa_2}(y)/\ell$	$s^2 = C_{\kappa_1}(x)S_{\kappa_1\kappa_2}(b)/\ell$	$s^2 = S_{\kappa_1}(r)S_{\kappa_2}(\phi)/\ell$
$\tilde{s}^+ = \dfrac{1 - \eta_+\{1 - C_{\kappa_1}(a)C_{\kappa_1\kappa_2}(y)\}}{1 + \eta_-\{1 - C_{\kappa_1}(a)C_{\kappa_1\kappa_2}(y)\}}$	$\tilde{s}^+ = \dfrac{1 - \eta_+\{1 - C_{\kappa_1}(x)C_{\kappa_1\kappa_2}(b)\}}{1 + \eta_-\{1 - C_{\kappa_1}(x)C_{\kappa_1\kappa_2}(b)\}}$	$\tilde{s}^+ = \dfrac{\ell^2 - T^2_{\kappa_1}(r/2)}{\ell^2 + T^2_{\kappa_1}(r/2)}$
$\tilde{s}^1 = \dfrac{S_{\kappa_1}(a)C_{\kappa_1\kappa_2}(y)/\ell}{1 + \eta_-\{1 - C_{\kappa_1}(a)C_{\kappa_1\kappa_2}(y)\}}$	$\tilde{s}^1 = \dfrac{S_{\kappa_1}(x)/\ell}{1 + \eta_-\{1 - C_{\kappa_1}(x)C_{\kappa_1\kappa_2}(b)\}}$	$\tilde{s}^1 = \dfrac{2\ell T_{\kappa_1}(r/2)C_{\kappa_2}(\phi)}{\ell^2 + T^2_{\kappa_1}(r/2)}$
$\tilde{s}^2 = \dfrac{S_{\kappa_1\kappa_2}(y)/\ell}{1 + \eta_-\{1 - C_{\kappa_1}(a)C_{\kappa_1\kappa_2}(y)\}}$	$\tilde{s}^2 = \dfrac{C_{\kappa_1}(x)S_{\kappa_1\kappa_2}(b)/\ell}{1 + \eta_-\{1 - C_{\kappa_1}(x)C_{\kappa_1\kappa_2}(b)\}}$	$\tilde{s}^2 = \dfrac{2\ell T_{\kappa_1}(r/2)S_{\kappa_2}(\phi)}{\ell^2 + T^2_{\kappa_1}(r/2)}$
$\tan A = \dfrac{T_{\kappa_1}(a)}{\ell}\dfrac{1 - \kappa_1 T^2_{\kappa_1}(r/2)}{1 - \frac{1}{\ell^2}T^2_{\kappa_1}(r/2)}$	$\sin X = \dfrac{S_{\kappa_1}(x)}{\ell}\dfrac{1 + \kappa_1 T^2_{\kappa_1}(r/2)}{1 + \frac{1}{\ell^2}T^2_{\kappa_1}(r/2)}$	$\tan^2(R/2) = \dfrac{1}{\ell^2}T^2_{\kappa_1}(r/2)$
$S_{\kappa_2}(Y) = \dfrac{S_{\kappa_1\kappa_2}(y)}{\ell}\dfrac{1 + \kappa_1 T^2_{\kappa_1}(r/2)}{1 + \frac{1}{\ell^2}T^2_{\kappa_1}(r/2)}$	$T_{\kappa_2}(B) = \dfrac{T_{\kappa_1\kappa_2}(b)}{\ell}\dfrac{1 - \kappa_1 T^2_{\kappa_1}(r/2)}{1 - \frac{1}{\ell^2}T^2_{\kappa_1}(r/2)}$	$\Phi = \phi$

4. Conformal embedding $S^2{}_{[\kappa_1],\kappa_2} \longmapsto C^2{}_{[\kappa_1],\kappa_2}$

The CK space $S^2_{[\kappa_1],\kappa_2}$ is naturally embedded into its conformal compactification $C^2_{[\kappa_1],\kappa_2}$. To describe this, a *group theoretical* procedure is the following: let us consider a point $Q \in S^2_{[\kappa_1],\kappa_2}$ with geodesic coordinates (a, y), (x, b) or (r, ϕ) [1]; this means that under the CK matrix realization, Q is obtained as the image of the origin point by the products $\exp(aP_1)\exp(yP_2)$, $\exp(bP_2)\exp(xP_1)$ or $\exp(\phi J_{12})\exp(rP_1)$, with these group elements computed in the 3×3 standard realization of the CK group, and *not* in the *conformal* 4×4 realization (2.2). By definition, the image of Q under the conformal embedding is the point $\tilde{Q}$, with coordinates $\mathbf{s} = (s^+, s^-, s^1, s^2)$, obtained from the origin $\mathcal{O} = (1, -1, 0, 0) \in \Gamma_0$ under the corresponding pairs of one-parameter subgroups, *now taken in the conformal realization* (2.2) yet with the same values for the canonical parameters. In this way the orbit of the *conformal origin* $\mathcal{O} \in \Gamma_0$ under the *initial CK group*, is identified as a subset of the conformal completion of the initial space, and the corresponding identification describes the *conformal embedding* from the former into the latter.

As the conformal completion itself has been identified first as a subset of the projective cone Γ_0, and next as a space $\widetilde{S}^2_{[+],\kappa_2}$ obtained by antipodal identification of points at infinity in a CK space, we have at hand another parametric description of the latter in terms either of the *conformal Weierstrass coordinates* or *conformal geodesic coordinates*, as given in (3.4). This leads to explicit relations between the usual geodesic coordinates (a, y), (x, b) or (r, ϕ) in the initial space and either the conformal geodesic coordinates (A, Y), (X, B), (R, Φ) or the conformal Weierstrass coordinates $\tilde{\mathbf{s}} = (\tilde{s}^+, \tilde{s}^1, \tilde{s}^2)$ in the conformal completion (recall $\tilde{s}^i = -s^i/s^-$ where s^+, s^-, s^i are the canonical coordinates in the conformal ambient space). The final results are displayed in table 2.

This embedding has also a very neat geometrical description as a *stereographic projection*. On the one hand, the initial CK space $S^2_{[\kappa_1],\kappa_2}$ can be realized in a linear ambient space $\mathbb{R}^3 = (x^0, x^1/\ell, x^2/\ell)$, with equation

$$(x^0)^2 + \kappa_1\ell^2(x^1/\ell)^2 + \kappa_1\kappa_2\ell^2(x^2/\ell)^2 = 1 \tag{4.1}$$

which can be considered as the CK space $S^2_{[\kappa_1\ell^2],\kappa_2}$. On the other hand, the corresponding conformal space $C^2_{[\kappa_1],\kappa_2} = \widetilde{S}^2_{[+],\kappa_2}$ is realized in the ambient space $\mathbb{R}^3 = (\tilde{s}^+, \tilde{s}^1, \tilde{s}^2)$ (the section $s^- = -1$ of the projective cone Γ_0) through (3.3). If both ambient spaces are identified as

$$\tilde{s}^+ \leftrightarrow x^0 \qquad \tilde{s}^1 \leftrightarrow x^1/\ell \qquad \tilde{s}^2 \leftrightarrow x^2/\ell \tag{4.2}$$

the former embedding of $S^2_{[\kappa_1],\kappa_2}$ into $C^2_{[\kappa_1],\kappa_2}$ turns out to *coincide* with a stereographic projection of $S^2_{[\kappa_1\ell^2],\kappa_2}$ into $\widetilde{S}^2_{[+],\kappa_2}$ with pole $\mathcal{P} = (-1, 0, 0)$:

$$(x^0 + 1, x^1/\ell, x^2/\ell) = \mu(\tilde{s}^+ + 1, \tilde{s}^1, \tilde{s}^2) \tag{4.3}$$

where the real proportionality factor μ is

$$\mu = \frac{2}{(1+\tilde{s}^+) + \kappa_1\ell^2(1-\tilde{s}^+)} = \frac{x^0+1}{2} + \frac{1-x^0}{2\kappa_1\ell^2}. \tag{4.4}$$

Hence the stereographic projection equations turn out to be

$$\begin{aligned}
\tilde{s}^+ &= \frac{\ell^2 - \frac{1}{\kappa_1}\left(\frac{1-x^0}{1+x^0}\right)}{\ell^2 + \frac{1}{\kappa_1}\left(\frac{1-x^0}{1+x^0}\right)} \qquad & x^0 &= \frac{(1+\tilde{s}^+) - \kappa_1\ell^2(1-\tilde{s}^+)}{(1+\tilde{s}^+) + \kappa_1\ell^2(1-\tilde{s}^+)} \\
\tilde{s}^i &= \frac{2\ell\frac{1}{1+x^0}}{\ell^2 + \frac{1}{\kappa_1}\left(\frac{1-x^0}{1+x^0}\right)}x^i \qquad & x^i &= \frac{2\ell}{(1+\tilde{s}^+) + \kappa_1\ell^2(1-\tilde{s}^+)}\tilde{s}^i \qquad i = 1, 2.
\end{aligned} \tag{4.5}$$

By substituting in the above expressions the parametrizations of (x^0, x^1, x^2) in terms of geodesic coordinates of $S^2_{[\kappa_1],\kappa_2}$ given in [1], we recover the results written in table 2 for the $\tilde{s}^i$ coordinates of $C^2_{[\kappa_1],\kappa_2}$. Such equations show the identification of antipodal pairs of points at infinity in $S^2_{[+],\kappa_2}$ because these are mapped by stereographical projection into the same image.

Thus we find the description of the embedding $S^2_{[\kappa_1],\kappa_2} \longmapsto C^2_{[\kappa_1],\kappa_2}$ in any coordinate system by identifying the parametrizations of $(\tilde{s}^+, \tilde{s}^1, \tilde{s}^2)$ just obtained with those given in conformal geodesic coordinates (3.4); these results are also collected in table 2. Such general expressions are illustrated in table 3 for the nine spaces in parallel I coordinates, $(a, y) \mapsto (A, Y)$, and represented in figure 1. Note that

$$\begin{aligned}
C_{\kappa_1}(a) &= C_{\kappa_1\ell^2}(a/\ell) \qquad & S_{\kappa_1}(a) &= \ell S_{\kappa_1\ell^2}(a/\ell) \\
C_{\kappa_1\kappa_2}(y) &= C_{\kappa_1\kappa_2\ell^2}(y/\ell) \qquad & S_{\kappa_1\kappa_2}(y) &= \ell S_{\kappa_1\kappa_2\ell^2}(y/\ell).
\end{aligned} \tag{4.6}$$

The embedding has a non-canonical character, as it depends on the choice of ℓ. Only the *sign* of κ_1 matters, as for any fixed κ_1, a suitable choice of ℓ can reduce the dimensionless product $\kappa_1\ell^2$ to either $1, 0, -1$.

Now we discuss separately the conformal embedding for each particular CK space.

4.1. Riemannian spaces: $S^2_{[\kappa_1],+} \longmapsto C^2_{[\kappa_1],+} \equiv \mathbf{S}^2$

The embedding of the *sphere* $S^2_{[+],+}$ covers the full sphere $\mathbf{S}^2$ once, so that $\mathbf{S}^2$ coincides with its conformal compactification.

Table 3. The conformal embedding in parallel I coordinates for the nine CK spaces: $(a, y) \equiv (t, y) \in S^2_{[\kappa_1],\kappa_2} \longmapsto (A, Y) \equiv (T, Y) \in C^2_{[\kappa_1],\kappa_2} \equiv \widetilde{S}^2_{[+],\kappa_2}$. The length ℓ is chosen in such a manner that $\kappa_1\ell^2 \in \{1, 0, -1\}$ and $\kappa_2 \in \{1, 0, -1/c^2\}$. In the six spacetimes, t is the time coordinate and y the space coordinate.

$\mathbf{S}^2 \longmapsto \mathbf{S}^2$	$\mathbf{E}^2 \longmapsto \mathbf{S}^2$	$\mathbf{H}^2 \longmapsto \mathbf{S}^2$
$\kappa_1\ell^2 = 1, \kappa_2 = 1$	$\kappa_1\ell^2 = 0, \kappa_2 = 1$	$\kappa_1\ell^2 = -1, \kappa_2 = 1$
$\tilde{s}^+ = \cos(a/\ell)\cos(y/\ell)$	$\tilde{s}^+ = \dfrac{4 - \{(a/\ell)^2 + (y/\ell)^2\}}{4 + \{(a/\ell)^2 + (y/\ell)^2\}}$	$\tilde{s}^+ = \dfrac{1}{\cosh(a/\ell)\cosh(y/\ell)}$
$\tilde{s}^1 = \sin(a/\ell)\cos(y/\ell)$	$\tilde{s}^1 = \dfrac{4a/\ell}{4 + \{(a/\ell)^2 + (y/\ell)^2\}}$	$\tilde{s}^1 = \tanh(a/\ell)$
$\tilde{s}^2 = \sin(y/\ell)$	$\tilde{s}^2 = \dfrac{4y/\ell}{4 + \{(a/\ell)^2 + (y/\ell)^2\}}$	$\tilde{s}^2 = \dfrac{\tanh(y/\ell)}{\cosh(a/\ell)}$
$\tan A = \tan(a/\ell)$	$\tan A = \dfrac{4a/\ell}{4 - \{(a/\ell)^2 + (y/\ell)^2\}}$	$\tan A = \sinh(a/\ell)\cosh(y/\ell)$
$\sin Y = \sin(y/\ell)$	$\sin Y = \dfrac{4y/\ell}{4 + \{(a/\ell)^2 + (y/\ell)^2\}}$	$\sin Y = \dfrac{\tanh(y/\ell)}{\cosh(a/\ell)}$

$\mathbf{NH}^{1+1}_+ \longmapsto \widetilde{\mathbf{NH}}^{1+1}_+$	$\mathbf{G}^{1+1} \longmapsto \widetilde{\mathbf{NH}}^{1+1}_+$	$\mathbf{NH}^{1+1}_- \longmapsto \widetilde{\mathbf{NH}}^{1+1}_+$
$\kappa_1\ell^2 = 1, \kappa_2 = 0$	$\kappa_1\ell^2 = 0, \kappa_2 = 0$	$\kappa_1\ell^2 = -1, \kappa_2 = 0$
$\tilde{s}^+ = \cos(t/\ell)$	$\tilde{s}^+ = \dfrac{4 - (t/\ell)^2}{4 + (t/\ell)^2}$	$\tilde{s}^+ = \dfrac{1}{\cosh(t/\ell)}$
$\tilde{s}^1 = \sin(t/\ell)$	$\tilde{s}^1 = \dfrac{4t/\ell}{4 + (t/\ell)^2}$	$\tilde{s}^1 = \tanh(t/\ell)$
$\tilde{s}^2 = y/\ell$	$\tilde{s}^2 = \dfrac{4y/\ell}{4 + (t/\ell)^2}$	$\tilde{s}^2 = \dfrac{y/\ell}{\cosh(t/\ell)}$
$\tan T = \tan(t/\ell)$	$\tan T = \dfrac{4t/\ell}{4 - (t/\ell)^2}$	$\tan T = \sinh(t/\ell)$
$Y = y/\ell$	$Y = \dfrac{4y/\ell}{4 + (t/\ell)^2}$	$Y = \dfrac{y/\ell}{\cosh(t/\ell)}$

$\mathbf{AdS}^{1+1} \longmapsto \widetilde{\mathbf{AdS}}^{1+1}$	$\mathbf{M}^{1+1} \longmapsto \widetilde{\mathbf{AdS}}^{1+1}$	$\mathbf{dS}^{1+1} \longmapsto \widetilde{\mathbf{AdS}}^{1+1}$
$\kappa_1\ell^2 = 1, \kappa_2 = -1/c^2$	$\kappa_1\ell^2 = 0, \kappa_2 = -1/c^2$	$\kappa_1\ell^2 = -1, \kappa_2 = -1/c^2$
$\tilde{s}^+ = \cos(t/\ell)\cosh(y/c\ell)$	$\tilde{s}^+ = \dfrac{4 - \{(t/\ell)^2 - (y/c\ell)^2\}}{4 + \{(t/\ell)^2 - (y/c\ell)^2\}}$	$\tilde{s}^+ = \dfrac{1}{\cosh(t/\ell)\cos(y/c\ell)}$
$\tilde{s}^1 = \sin(t/\ell)\cosh(y/c\ell)$	$\tilde{s}^1 = \dfrac{4t/\ell}{4 + \{(t/\ell)^2 - (y/c\ell)^2\}}$	$\tilde{s}^1 = \tanh(t/\ell)$
$\tilde{s}^2 = c\sinh(y/c\ell)$	$\tilde{s}^2 = \dfrac{4y/\ell}{4 + \{(t/\ell)^2 - (y/c\ell)^2\}}$	$\tilde{s}^2 = \dfrac{c\tan(y/c\ell)}{\cosh(t/\ell)}$
$\tan T = \tan(t/\ell)$	$\tan T = \dfrac{4t/\ell}{4 - \{(t/\ell)^2 - (y/c\ell)^2\}}$	$\tan T = \sinh(t/\ell)\cos(y/c\ell)$
$\sinh(Y/c) = \sinh(y/c\ell)$	$\sinh(Y/c) = \dfrac{4(y/c\ell)}{4 + \{(t/\ell)^2 - (y/c\ell)^2\}}$	$\sinh(Y/c) = \dfrac{\tan(y/c\ell)}{\cosh(t/\ell)}$

For the *Euclidean* plane $\mathbf{E}^2$, the embedding covers the full sphere $\mathbf{S}^2$ minus a single point, with conformal Weierstrass coordinates $(-1, 0, 0)$, that is, the pole $\mathcal{P}$ of the stereographic projection. This is the usual point at infinity required to globally define inversions and transforms $\mathbf{E}^2$ into its 1-point compactification, the Riemann sphere.

The embedding of the *Lobachewski* plane $\mathbf{H}^2$ only covers a disc in the sphere (when ℓ is chosen such that $\kappa_1\ell^2 = -1$, this disc is exactly the half-sphere given by $\tilde{s}^+ > 0$), and points at infinity in $\mathbf{H}^2$ appear as ordinary points in the disc boundary, which for $\kappa_1\ell^2 = -1$ is the equator ($\tilde{s}^+ = 0$); the new points added in the compactification make up the complementary

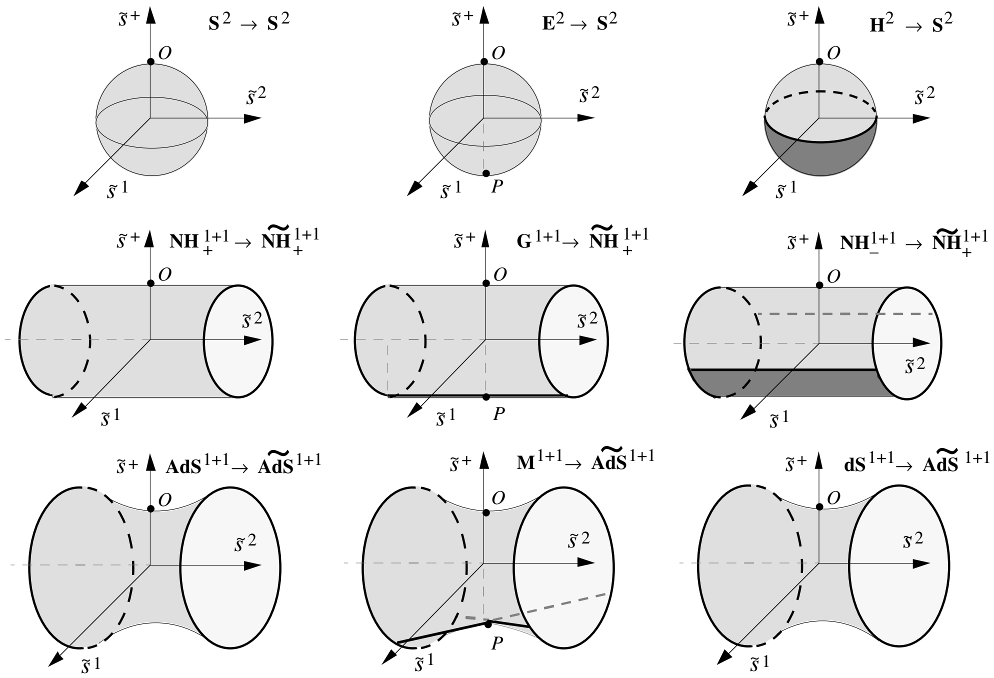

Figure 1. Visualization of the conformal compactification $C^2_{[\kappa_1],\kappa_2} \equiv \tilde{S}^2_{[+],\kappa_2}$, for the nine CK spaces in conformal Weierstrass coordinates $(\tilde{s}^+, \tilde{s}^1, \tilde{s}^2)$; the point $\mathcal{O} = (1, 0, 0)$ is the origin in $C^2_{[\kappa_1],\kappa_2}$ and $\mathcal{P} = (-1, 0, 0)$ is its antipodae. Marked points, lines or regions in darker grey indicate the elements which should be added to the initial space in order to obtain its compactification. For the six spacetimes with $\kappa_2 \leqslant 0$, there is antipodal identification of points at infinity corresponding to the two circles $(\tilde{s}^+)^2 + (\tilde{s}^1)^2 = \text{constant}$.

disc ($\tilde{s}^+ < 0$ if $\kappa_1 \ell^2 = -1$), which may be considered as *another copy of* $\mathbf{H}^2$ glued to the proper embedding of the hyperbolic plane in $\mathbf{S}^2$ by the points in the discs' boundaries (the equator $\tilde{s}^+ = 0$ which plays the role of the points at infinity in both copies of the initial space $\mathbf{H}^2$). In this structure we can recognize the half-plane model of hyperbolic space, whose other half contains naturally another copy of $\mathbf{H}^2$.

4.2. Non-relativistic spacetimes: $S^2_{[\kappa_1],0} \longmapsto C^2_{[\kappa_1],0} \equiv \widetilde{\mathbf{NH}}_+^{1+1}$

The transition from $\mathbf{NH}_+^{1+1}$ to $\widetilde{\mathbf{NH}}_+^{1+1}$ requires antipodal identification of the two circles $(\tilde{s}^+)^2 + (\tilde{s}^1)^2 = 1$ at the infinity $\tilde{s}^2 = \pm\infty$. Thus the new space is compact. Once this is assumed, the description of the compactification of the three non-relativistic spacetimes is straightforward. For *oscillating NH* spacetime $\mathbf{NH}_+^{1+1}$, one has simply to embed identically $\mathbf{NH}_+^{1+1}$ into itself.

For the *Galilei* spacetime $\mathbf{G}^{1+1}$, its conformal embedding into $\mathbf{NH}_+^{1+1}$ covers this space minus the line $\tilde{s}^+ = -1$, $\tilde{s}^1 = 0$, any $\tilde{s}^2$; the (pole) point $\mathcal{P} = (-1, 0, 0)$ on this line plays the role of the point at infinity in $\mathbf{G}^{1+1}$ and other points on this line correspond to the 'instantaneous 1D space' of the point at infinity.

The conformal embedding of the *expanding NH* spacetime $\mathbf{NH}_-^{1+1}$ into $\mathbf{NH}_+^{1+1}$ covers one half of the cylinder (say the 'upper' one $\tilde{s}^+ > 0$), so that one has to add as new points the remaining 'lower' half of the cylinder ($\tilde{s}^+ < 0$) glued to the other proper half by the two lines in the plane $\tilde{s}^+ = 0$.

4.3. Relativistic spacetimes: $S^2_{[\kappa_1],-} \longmapsto C^2_{[\kappa_1],-} \equiv \widetilde{\mathbf{AdS}}^{1+1}$

In these cases we set $\kappa_2 = -1/c^2$. The transition from $\mathbf{AdS}^{1+1}$ to $\widetilde{\mathbf{AdS}}^{1+1}$ follows by antipodal identification of the two circles at the *spatial infinity* in the *anti-de Sitter* spacetime: $(\tilde{s}^+)^2 + (\tilde{s}^1)^2 - (\tilde{s}^2)^2/c^2 = 1$, that is, $(\tilde{s}^+)^2 + (\tilde{s}^1)^2 = \infty$ for $\tilde{s}^2 = \pm\infty$; each circle $\tilde{s}^2 = \infty$, $\tilde{s}^2 = -\infty$ is antipodal to the other. The compact time-like lines (label $\kappa_1 > 0$) embed homeomorphically, while the space-like lines, whose label is $\kappa_1\kappa_2 < 0$, hence hyperbolic and originally not compact, are glued by their points at infinity with their antipodal lines in $\mathbf{AdS}^{1+1}$, following the antipodal identification of their points at infinity; hence they also become compact. Topologically the space so obtained is $\mathbf{S}^1 \times \mathbf{S}^1$. One $\mathbf{S}^1$ corresponds to the originally compact time-like line l_1. The other comes from the 1D compactification of the originally hyperbolic, hence non-compact space-like line l_2, which is obtained by glueing two copies of a hyperbolic line (l_2 and its antipode) by their points at infinity.

For *Minkowskian* spacetime $\mathbf{M}^{1+1}$, the completion requires adding two lines with equations $\tilde{s}^+ = -1$, $\tilde{s}^2 = \pm c\tilde{s}^1$ crossing through the intersection (pole) point $\mathcal{P}$. This is rather well known (compare, e.g., [12]): the new point $\mathcal{P}$ corresponds to the point 'at infinity' in $\mathbf{M}^{1+1}$ and the two new lines are the light-cone of the point at infinity. We stress that this description appears as a particular case within the complete scheme, and its dependence on curvature can be clearly seen.

Finally, the *de Sitter* spacetime $\mathbf{dS}^{1+1}$, with hyperbolic non-compact time-like lines ($\kappa_1 < 0$) and compact space-like ones ($\kappa_1\kappa_2 > 0$) is the only CK space where parallel I coordinates do not completely cover the space, so the expressions in table 3 do not provide a description in all $\mathbf{dS}^{1+1}$. In this case it is better to use stereographic projection directly, which maps the whole de Sitter spacetime into $\mathbf{AdS}^{1+1}$ in a one-to-one way compatible with antipodal identification in $\mathbf{AdS}^{1+1}$. This could have been foreseen, as essentially $\mathbf{dS}^{1+1}$ and $\mathbf{AdS}^{1+1}$ are the same space, with an interchange time-like $\leftrightarrow$ space-like, so we can expect $\mathbf{dS}^{1+1}$ and $\mathbf{AdS}^{1+1}$ to have essentially the same compactification. If we are interested only in the double wedge in $\mathbf{dS}^{1+1}$ covered by parallel I coordinates (with focal points at the poles of the initial time-like line l_1), the embedding of this region is determined by $(\tilde{s}^1)^2 < 1$, that is, $0 < (\tilde{s}^+)^2 - (\tilde{s}^2)^2/c^2 \leqslant 1$, which is limited by four lines with equations $\tilde{s}^1 = \pm 1$, $\tilde{s}^2 = \pm c\tilde{s}^+$; these four lines are the two pairs of isotropic lines through the two poles of the initial non-compact time-like line l_1. However, the other regions limited by these lines in the conformal compactification are not new points, but the images by the embedding of the region of $\mathbf{dS}^{1+1}$ not covered by the parallel I coordinates.

5. Concluding remarks

It is well known that the action of the isometry (kinematical) group $SO_{\kappa_1,\kappa_2}(3)$ on the 2D spaces $S^2_{[\kappa_1],\kappa_2}$ can be linearized in an ambient space $\mathbb{R}^3$ with one extra dimension. In this paper we have explicitly shown that for any constant curvature or metric signature type, the conformal group $\mathrm{conf}_{\kappa_1,\kappa_2}$ can be realized as a matrix group acting as *globally defined* linear transformations in a 'conformal ambient space' $\mathbb{R}^4$, with *two* extra dimensions, and thus acting (non-linearly but still globally) in a suitable *conformal* extension of the initial space. By pursuing this construction within the CK viewpoint, the present paper affords a new approach to conformal groups that produces a very explicit description of the conformal compactification of homogeneous spacetimes. The result that the conformal completion of a space of constant curvature κ_1 does not depend on the curvature may be foreseen from the well known fact that every 2D metric, constant curvature or not, is conformally flat, and hence *a fortiori* the conformal compactification of a curved space will coincide with that of its corresponding flat one. This can also be clearly seen in our approach, but we have obtained much more than that, including a joint view of the structure of these conformal compactifications that allows a clear understanding and visualization of how the embedding of the initial spacetime changes when its curvature vanishes or when the metric degenerates.

Finally we stress that the structure of conformal compactification of Minkowskian spacetime appears as a particular instance within our parametric approach. This issue is of relevance in view of the current interest in AdS–CFT correspondence as a conjecture relating local QFT on $\mathbf{AdS}^{1+(d-1)}$ to a conformal QFT on the compactified Minkowski spacetime comp $\mathbf{M}^{1+(d-2)}$ (see [15] and references therein). In this context an explicit description of the geometry behind these relativistic spacetimes in a way as general as possible should be helpful.

Acknowledgments

This work was partially supported by the Ministerio de Ciencia y Tecnología, Spain (Projects PB98-0370 and BFM2000-1055).

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